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W. M. HIRSCH, J. CONN, AND C. SIEGEL

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1. Introduction

We consider in this paper a queueing process that was encountered in the course of a military study. Imagine the following situation: A missile battery is defending itself against a train of hostile aircraft. Each attacking plane has perfect accuracy and consequently will destroy the battery if it gets within a certain critical range[(1)]. We think of the file of aircraft as a queue which is moving toward an objective 0 and of the battery as a server who attempts to perform some operation on each element in the queue before that element reaches 0. When a member of the queue has been served successfully, the server retracts instantly to the next one in line. If an element in the queue reaches 0 without having previously been served, the server is permanently and totally disabled.

A similar problem was formulated by B. McMillan and J. Riordan [3]. In their treatment the time axis is regarded as continuous, and the service time is assumed to be an exponentially distributed random variable. Since, in the aforementioned example, each launching of a missile renders the launching facility inoperable for some fixed period of time, independently of the success or failure of the firing, it is of interest to study the process under the assumption that commencement and completion of service can occur only at definite, regularly spaced time points. Accordingly, in this paper time is treated as a discrete variable.

[(1)] This idealization is made to simplify the model. The more realistic case, when arrival within critical range results in disability of the battery with probability α, $0 \leq \alpha \leq 1$, is treated by W. Hirsch and H. Hanish in "Some Properties of a Transient Queue" (to appear).

2. The Mathematical Model

Let E_1, E_2, ... denote, respectively, members of an infinite queue, where the subscript indicates the order in line. We think of these elements as being located, initially, at a subset of the integer lattice points on the positive X-axis, the precise position of each element being stochastically determined. Specifically, let ξ_0 be a random variable which takes the value k with probability a_k, $\sum_{k=1}^{\infty} a_k = 1$, and suppose that at time $t = 0$ (the initial instant) element E_1 is at the point ξ_0. The distance between E_j and E_{j+1} is a random variable ξ_j which, for all j, takes the value k with probability b_k, $k = 1,2,\ldots$. The random variables $\{\xi_n, n=0, 1,\ldots\}$ are assumed to be mutually independent.

The queue moves with unit velocity toward the origin 0, its members being served successively. Service on E_j is initiated instantaneously upon completion of service on E_{j-1}. If an element arrives at 0 unserviced, the server is disabled (absorbed) and the process terminates. (No interruption occurs if an element reaches 0 and simultaneously service on it is completed). The time between commencement and completion of service on E_j is a random variable τ_j which, for all j, takes the value k, $k = 1,2,\ldots$, with probability C_k. The sequence $\{\tau_n\}$ is mutually independent and, moreover, is independent of $\{\xi_n\}$. The above definitions imply that initiation and completion of service can occur only at integer lattice points.

3. Recurrence Relation for the Service Distribution

Let ν denote the number of elements served before termination of the process. Then ν is a (possibly improper) random variable and we set

$$P\{\nu = n\} = p_n \ ,$$

and

$$P\{\nu = n | \xi_o = k\} = p(n,k) \ .$$

Clearly,

(1) $$p_n = \sum_{k=1}^{\infty} p(n,k)a_k \ .$$

Let

$$q_k = 1 - \sum_{j=1}^{k} c_j \ .$$

Then, since $\nu = 0$ if and only if E_1 arrives at 0 unserviced, we have

(2) $$p(0,k) = q_k \ .$$

The random variable ν takes the value $n \geq 1$ if and only if service on E_1 is completed, and then exactly $n - 1$ additional elements are serviced prior to termination. To determine the probability of this event we argue conditionally, assuming that $\xi_o = k$ and $\xi_1 = m$. The probability of completing service on E_1 at the point j, $0 \leq j \leq k - 1$, is simply,

$$P\{\tau_1 = k - j\} = c_{k-j} \ .$$

If this occurs, the probability that exactly $n - 1$ additional elements are serviced prior to the absorption of the server is $p(n - 1, j + m)$. Hence,

$$p\{\nu = n | \xi_o = k, \xi_1 = m\} = \sum_{j=0}^{k-1} c_{k-j} p(n - 1, j + m) \quad ,$$

and therefore

$$(3) \qquad p(n,k) = \sum_{j=0}^{k-1} c_{k-j} \sum_{m=1}^{\infty} p(n - 1, j + m) b_m \; .$$

The recursion relation (3) and the initial condition (2) uniquely determine the conditional probabilities $p(n,k)$, $n = 0,1,\ldots$. The absolute probabilities $\{p_n\}$ can then be computed from (1).

4. <u>Uniform Spacing and Geometric Service Time</u>

4.1 <u>The exact solution</u>. Let us now assume that the time spent by the server on a given queue element is determined by a sequence of independent attempts at service, each taking unit time and having probability p $(0 < p < 1)$ of success. The service time τ_j is then geometrically distributed, i.e.,

$$c_k = pq^{k-1} \quad ,$$

where $q = 1 - p$. Suppose further that the distance ξ_j between consecutive elements in the queue has the degenerate distribution

$$P\left\{\xi_j = d\right\} = 1 \quad ,$$

where d is a positive integer.

The initial condition (2) becomes

$$p(0,k) = q_k = q^k \quad , \tag{4}$$

and the recursion relation (3) takes the form

$$p(n,k) = \sum_{j=0}^{k-1} pq^{k-j-1} p(n-1,\ j+d) \quad . \tag{5}$$

The unique solution of (5) under the constraint (4) is given in the following theorem:

Theorem 1. If the elements in the queue are spaced at equal intervals d, and the service time has the distribution $C_k = pq^{k-1}$, then

$$p(n,k) \equiv \frac{k(k+nd-1)!}{n![k+n(d-1)]!}\ p^n q^{k+n(d-1)} \quad . \tag{6}$$

Proof. The proof is inductive. It is evident from (4) and (5) that (6) holds for n = 1. Suppose it is true for n = r. Then, defining

$$\alpha^{(m)} = \frac{\alpha!}{(\alpha-m)!} \quad ,$$

we have

$$p(r+1,k) = \frac{p^{r+1}q^{k+(r+1)d-1)}}{r!}\sum_{j=0}^{k-1}(j+d)[j+(r+1)d-1]^{(r-1)}$$

$$= \frac{p^{r+1}q^{k+(r+1)(d-1)}}{r!}\sum_{j=0}^{k-1}\Big\{[j+(r+1)d]^{(r)}$$

$$- rd[j+(r+1)d-1]^{(r-1)}\Big\} .$$

Using the identity

$$\sum_{a=m}^{n} a^{(r)} = \frac{(n+1)^{(r+1)} - m^{(r+1)}}{r+1} ,$$

we find that

$$p(r+1,k) = \frac{p^{r+1}q^{k+(r+1)(d-1)}}{r!}\Big\{\frac{[k+(r+1)d][k+(r+1)d-1]^{(r)}}{r+1}$$

$$- d[k+(r+1)d-1]^{(r)}\Big\}$$

$$= \frac{p^{r+1}q^{k+(r+1)(d-1)}}{(r+1)!}\, k[k+(r+1)d-1]^{(r)} ,$$

which completes the induction.

4.2 Asymptotic behavior. The limiting behavior of p(n,k) as $n \longrightarrow \infty$ is obtained easily by using the Sterling approximation and the relation

$$(n+m)! \sim n!n^{m} .$$

For d = 1,

$$p(n,k) \sim \frac{q^k}{(k-1)!} n^{k-1} p^n ,$$

while for $d > 1$

$$p(n,k) \sim \frac{C\beta^n}{n^{3/2}} ,$$

where

$$C = k \left(\frac{qd}{d-1}\right)^k \frac{1}{\sqrt{2\pi d(d-1)}} ,$$

and

$$\beta = \frac{pq^{d-1}d^d}{(d-1)^{d-1}} .$$

The function $\beta = \beta(p)$, which takes values in the interval $0 \leq \beta \leq 1$ when $0 \leq p \leq 1$, is monotone increasing on $0 \leq p \leq 1/d$ and monotone decreasing on $1/d \leq p \leq 1$. Clearly,

$$\beta(\tfrac{1}{d}) = 1 .$$

Consequently, if $p < p^* \leq 1/d$, we have (in an obvious notation)

$$p(n,k) = o(p^*(n,k))$$

as $n \longrightarrow \infty$. If $\frac{1}{d} \leq p < p^*$,

$$p^*(n,k) = o(p(n,k)) .$$

On intuitive grounds we might have anticipated that for sufficiently large values of n,

$$p(n,k) < p^{*}(n,k) \quad .$$

Paradoxically, however, the opposite inequality holds in the range $\frac{1}{d} \leq p < 1$. The apparent paradox is resolved in the next section, where it is shown that there is a positive probability of servicing infinitely many elements if and only if $p > 1/d$, so that for values of p in this range some of the probability mass is shifted out to infinity.

4.3 <u>The probability of ultimate absorption.</u> In this section we study the behavior of the quantity

$$f = \sum_{n=0}^{\infty} p(n,k) \quad ,$$

which can be interpreted as the probability that the server is disabled ultimately, i.e., after serving some finite number of elements. (The difference 1-f is the probability that all members of the queue are served, or equivalently, that the server's life-span is infinite.) For this purpose it is convenient to introduce the generating function

$$P(x,k) = \sum_{n=0}^{\infty} p(n,k)x^{k} \quad ,$$

restricted to the domain $0 \leq x \leq 1$. The properties of P of interest to us can be obtained expeditiously by using the following characterization:

<u>Theorem 2.</u> If $p \neq \frac{1}{d}$, to each real number x in the closed interval $0 \leq x \leq 1$ there corresponds exactly one value $z = z(x)$ in the disc

$$|z-1| < \frac{1}{d-1}$$ [2]

such that

$$z = 1 + xpq^{d-1}z^d \quad . \tag{7}$$

The functions $P(x,k)$ and $z(x)$ are connected by the relation

$$P(x,k) = [qz(x)]^k \quad . \tag{8}$$

If $p = \frac{1}{d}$, the same result holds in the half-open interval $0 \leq x < 1$.

<u>Proof.</u> If $d = 1$, equation (7) has the unique solution

$$z(x) = \frac{1}{1-xp} \; ,$$

which by the binomial expansion, leads to

$$[qz(x)]^k = q^k(1-xp)^{-k} = q^k + \sum_{n=1}^{\infty} \frac{p^n q^k k(k+n-1)^{(n-1)}}{n!} x^n \quad .$$

Comparing this equation with (6), we see that for $d = 1$

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[2] If $d = 1$, the disc may be taken to be the infinite z-plane.

$$P(x,k) = [qz(x)]^k = \left(\frac{q}{1-xp}\right)^k .$$

Suppose now that $d > 1$. Let G denote the open disc

$$|z-1| < \frac{1}{d-1} ,$$

and let Γ be its boundary. The functions z^k and $\frac{z-1}{z^d}$ are analytic in $\overline{G}$, the closure of G, and it is easy to verify that

$$\inf_{z \varepsilon \Gamma} \left|\frac{z-1}{z^d}\right| = \frac{(d-1)^{d-1}}{d^d} .$$

Applying the theorem of Bürmann-Lagrange we conclude that to each point w in the disc

$$|w| < \frac{(d-1)^{d-1}}{d^d}$$

there corresponds a unique point

$$z = \phi(w)$$

in G such that

$$z = 1 + wz^d ,$$

and the function

$$z^k = \phi^k(w)$$

has the series representation

$$z^k = \phi^k(w) = 1 + \sum_{n=1}^{\infty} \frac{k(k+nd-1)^{(n-1)}}{n!} w^n \quad .$$

It follows from the inequality of the arithmetic and geometric means that if either $p \neq \frac{1}{d}$ or $x \neq 1$,

$$xpq^{d-1} < \frac{(d-1)^{d-1}}{d^d} \quad .$$

Hence we may replace w by xpq^{d-1} in the Bürmann-Lagrange expansion. Setting

$$\phi(xpq^{d-1}) = z(x)$$

and multiplying both sides by q^k we find that

$$(9) \quad [qz(x)]^k = q^k + \sum_{n=1}^{\infty} \frac{p^n q^{k+n(d-1)} k(k+nd-1)^{(n-1)}}{n!} x^n \quad ,$$

where $z(x)$ is the unique root in G of the equation

$$z = 1 + xpq^{d-1} z^d \quad .$$

We observe that the right-hand side of (9) is $P(x,k)$, which completes the proof.

Suppose now that $p = \frac{1}{d}$. Then the function $z(x)$ appearing in the preceding theorem is defined only on the half-open interval $0 \leq x < 1$. However, its definition can be extended to

the closed interval so that relations (7) and (8) are preserved by setting

$$z(1) = \lim_{z \longrightarrow 1^-} z(x) \quad . \tag{10}$$

The existence of this limit is guaranteed by the continuity and monotonicity of z. From Abel's theorem we conclude that

$$P(1,k) = [qz(1)]^k \quad .$$

Moreover, letting $x \longrightarrow 1^-$ in (7) it is clear that $z = z(1)$ satisfies the equation

$$z = 1 + pq^{d-1}z^d = \frac{1+(d-1)^{d-1}}{d^d} z^d \quad . \tag{11}$$

We cannot assert, however, that the value z(1) defined by (10) lies in the disc

$$|z-1| < \frac{1}{d-1} \quad .$$

Indeed it is easy to see that (11) has the unique real root

$$z = 1 + \frac{1}{d-1} \quad ,$$

which lies on the boundary of the disc, since the function

$$y(z) = \frac{z-1}{z^d}$$

increases monotonically in the interval $0 < z < 1 + \frac{1}{d-1}$, attains the value $\frac{(d-1)^{d-1}}{d^d}$ when $z = 1 + \frac{1}{d-1}$, and thereafter decreases monotonically.

We now use the representation of P given above to derive a necessary and sufficient condition that $f = 1$:

Theorem 3.[3] The probability f of ultimate termination of service has the value 1 if and only if $d \leq \frac{1}{p}$, i.e., the distance between adjacent elements in the queue does not exceed the expected service time.

Proof. If $1 \leq d < \frac{1}{p}$, the point $\frac{1}{q}$ lies in the disc $|z - 1| < \frac{1}{d-1}$. However, $z = \frac{1}{q}$ is a root of (7) when $x = 1$. Since there is but one root in the disc, we have

$$z(1) = 1/q \quad ,$$

and therefore

$$f = P(1,k) = [qz(1)]^k = 1 \quad .$$

If $d = \frac{1}{p}$, equation (11) has the unique real root $z(1) = \frac{1}{q}$ and, once again,

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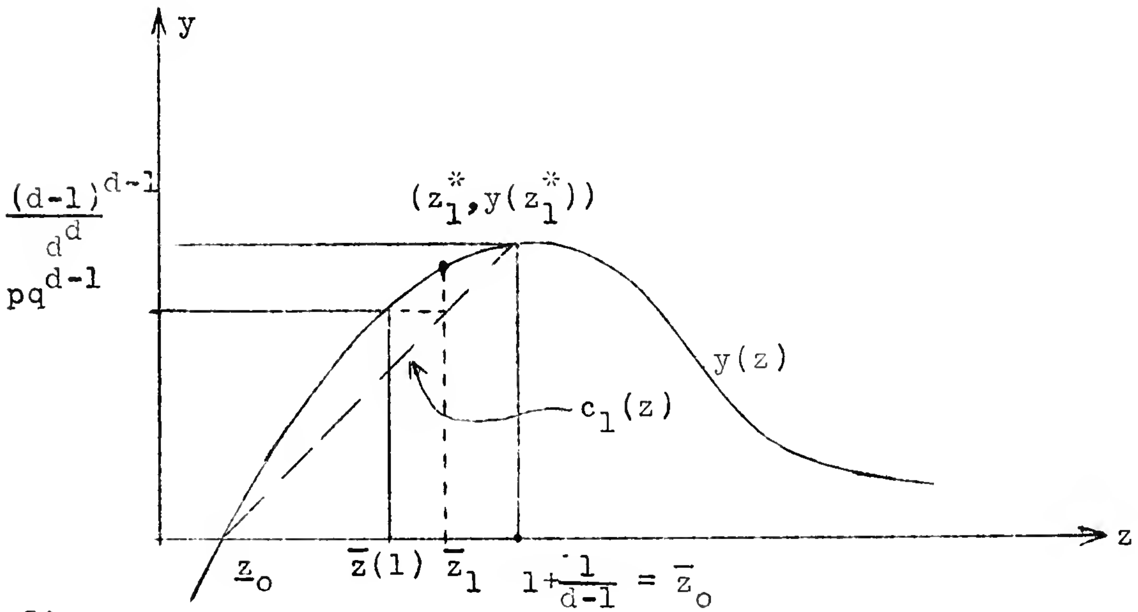
(3) After this theorem was obtained it was shown by W. Hirsch and H. Hanish in "Some Properties of a Transient Queue" (to appear) that it is a special case of a much more general result: In a wide class of queueing processes with an absorbing state ultimate termination of service is almost certain if and only if the expected distance between adjacent elements in the queue is not larger than the expected service time.

$$f = 1 \quad .$$

To complete the proof we note that if $d > \frac{1}{p}$, then $q < \frac{d-1}{d}$, and

$$f = [qz(1)]^k < [\frac{d-1}{d}(1 + \frac{1}{d-1})]^k = 1.$$

When $f < 1$, it is possible to estimate its magnitude. We do this by approximating $z(1)$.



Since

$$z(1) = \left[\frac{P(1,k)}{q^k}\right]^{\frac{1}{k}} = \left[1+q^{-k}\sum_{n=1}^{\infty} p(n,k)\right]^{\frac{1}{k}} > 1 \ ,$$

a first crude estimate is given by the interval

$$1 < z(1) < 1 + \frac{1}{d-1} \quad .$$

Now $y(z) = \frac{z-1}{z^d}$ is concave in this interval, and consequently the chord $C_1(z)$, connecting $(1,0)$ to $(1 + \frac{1}{d-1}, y(1 + \frac{1}{d-1}))$, lies completely below $y(z)$ in the interval. By solving the equation

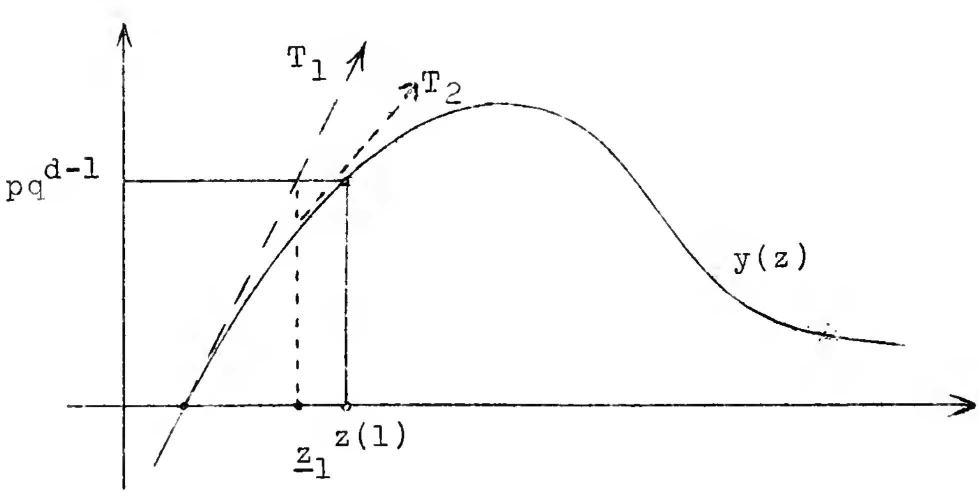
$$C_1(z) = pq^{d-1}$$

for z we obtain an improved upper bound for $z(1)$. Let us call this number $\bar{z}_1$. If we now use the chord $C_2(z)$, connecting $(1,0)$ to $(\bar{z}_1, y(\bar{z}_1))$, and repeat the described process, we obtain a further refinement $\bar{z}_2$. This iteration scheme yields a decreasing sequence $\{\bar{z}_n\}$ converging to $z(1)$. The values of $\{\bar{z}_n\}$ can be obtained recursively from the relation

$$\bar{z}_n = 1 + pq^{d-1}(\bar{z}_{n-1})^d, \quad n = 1,2,\ldots,$$

with

$$\bar{z}_o = 1 + \frac{1}{d-1} \quad .$$



We improve the lower bound for $z(1)$ by tangential approximation to the curve $y(z)$. A first improvement is obtained by constructing $T_1(z)$, the tangent line to $y(z)$ through the point $(1,0)$, and solving the equation

$$T_1(z) = pq^{d-1}$$

for z. Let $\underline{z}_1$ denote the solution. For further refinement we construct $T_2(z)$, the tangent line through $(\underline{z}_1, y(\underline{z}_1))$, and find $\underline{z}_2$ from the equation

$$T_2(z) = pq^{d-1} \quad .$$

This iteration process yields an increasing sequence $\{\underline{z}_n\}$ converging to $z(1)$. It is easy to verify that

$$\underline{z}_n = \frac{pq^{d-1} - \underline{z}_{n-1}^{-d}(\underline{z}_{n-1}-1)(1+d) + \underline{z}_{n-1}^{-d+1}}{\underline{z}_{n-1}^{-d} - d\underline{z}_{n-1}^{-d-1}(\underline{z}_{n-1}-1)}, \quad n = 1,2,\ldots,$$

with

$$\underline{z}_o = 1 \quad .$$

4.4 <u>Moments of the service distribution.</u> By successive differentiation of the equations

$$P(x,k) = [qs(x)]^k$$

and

$$z(x) = 1 + xpq^{d-1}z^d(x)$$

the moments of ν, the number of elements served before termination, can be calculated. The first two of these are given in the following theorem:

Theorem 4. The expected number of elements serviced, $E[\nu]$, is finite if and only if $d < \frac{1}{p}$. In this case

$$E[\nu] = \frac{kp}{1-dp}$$

and

$$\sigma^2[\nu] = \frac{kpq}{(1-dp)^3} \quad .$$

Proof. Using the algebraic equation satisfied by $z(x)$ we find that

$$z'(x) = \frac{pq^{d-1}z^d(x)}{1-xpq^{d-1}dz^{d-1}(x)} \quad .$$

Since

$$P'(x,k) = kq^k z^{k-1}(x)z'(x) \quad ,$$

we have

(12) $$P'(x,k) = \frac{kq^{k+d-1}z^{k+d-1}(x)}{1-xpq^{d-1}dz^{d-1}(x)} \quad .$$

It was shown in the proof of Theorem 3 that $z(1) = 1/q$ if $d \leq \frac{1}{p}$. Setting $x = 1$ in (12) it follows that

$$E[\nu] = p'(1,k) = \frac{kp}{1-dp} \quad .$$

If $d > \frac{1}{p}$, Theorem 3 shows that $P\left\{ \nu = \infty \right\} > 0$, and hence ν has infinite expectation. Thus, the expected value is finite if and only if $d < \frac{1}{p}$.

The variance is calculated by a second differentiation. For $d < \frac{1}{p}$ we find that

$$P''(1,k) = \frac{kp^2}{(1-dp)^2}\left(k + d - \frac{1-d}{1-dp}\right) \quad .$$

Hence,

$$\sigma^2[\nu] = p''(1,k) + p'(1,k) - [p'(1,k)]^2 = \frac{kpq}{(1-dp)^3} \quad .$$

5. Geometric Spacing and Geometric Service time.

5.1 The Generating Function

In this section, as in the previous one, we assume the service time to be geometrically distributed with parameter p $(0 < p < 1)$, so that

$$C_k = pq^{k-1} \ , \quad k = 1,2,\ldots \quad .$$

The distance between adjacent elements, however, is no longer taken to be uniform. We assume, rather, that it is determined by a succession of Bernoulli trials, each having probability $\tilde{p}$ $(0< \tilde{p} < 1)$ of success. The distance takes the value k if the first success occurs at the kth trial. Hence,

$$b_k = \tilde{p}\tilde{q}^{k-1} \ , \quad k = 1,2,\ldots \quad .$$

Although in principle the probabilities $\{p(n,k)\}$ can be calculated recursively from (2) and (3), it is difficult to obtain a general expression in this way. On the other hand, the explicit form of the generating function $P(x,k)$ can be obtained with comparative ease. Using (2) and (3) we find that $P(x,k)$ satisfies the equation:

$$(13) \qquad P(x,k) = q^k + \tilde{p}px \sum_{j=0}^{k-1} q^{k-j-1} \sum_{m=1}^{\infty} P(x,j+m)\tilde{q}^{m-1} \quad .$$

Setting $x = 0$, we have

$$(14) \qquad P(0,k) = q^k \quad .$$

Moreover, successive differentiation determines the values $\frac{d^nP}{dx^n}$ at $x = 0$. Hence (13) has only one solution which is analytic in the unit disc.

If we substitute a trial solution of the form

$$P(x,k) = U^k(x) \ ,$$

we are led to a quadratic equation for U:

$$\tilde{q}U^2 + (\tilde{p}px - 1 - \tilde{q}q)U + q = 0 \quad .$$

The root satisfying the condition $U(0) = q$ is

$$(15) \qquad U_1(x) = \frac{1 + q\tilde{q} - p\tilde{p}x - [(1-\tilde{q}q)^2 - 2p\tilde{p}(1+q\tilde{q})x + (p\tilde{p})^2x^2]^{1/2}}{2\tilde{q}} \;.$$

Since this function is analytic in the unit disc and satisfies (13), it follows that

$$(16) \qquad P(x,k) = U_1^k(x) \;.$$

Putting $x = 1$ in (15) we see that

$$U_1(1) = \frac{\tilde{q} + q - |\tilde{q} - q|}{2\tilde{q}} \;,$$

which, together with (16), establishes the following theorem:

<u>Theorem 5.</u>[(4)] The probability $f = P(1,k)$ of ultimate termination of service has the value 1 if and only if $\tilde{q} \leq q$, i.e., the expected distance between adjacent elements does not exceed the expected service time. If $\tilde{q} > q$,

$$f = \left(\frac{q}{\tilde{q}}\right)^k \;.$$

The moments of ν are readily calculated from (15) and (16). We give the first two of these below:

<u>Theorem 6.</u> The expected number of elements served is finite if and only if $\tilde{p} > p$. In this case

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(4) This is another instance of the general situation described in the footnote on p. 13.

$$E[\nu] = P'(1,k) = \frac{kp\tilde{p}}{\tilde{p}-p} \quad ,$$

and

$$\sigma^2[\nu] = P''(1,k) + P'(1,k) - [P'(1,k)]^2 = \frac{kp\tilde{p}}{(\tilde{p}-p)^3}(\tilde{p}^2 q + p^2\tilde{q})$$

5.2 Asymptotic behavior. The behavior of the probabilities $\{p(n,k)\}$ for large values of n is described in the following theorem:

Theorem 7. As $n \longrightarrow \infty$,

$$p(n,k) \sim \frac{Cr^n}{n^{3/2}}\text{(5)} \quad ,$$

where C is a constant (depending on k), and

$$r = \frac{p\tilde{p}(1+q\tilde{q}-2\sqrt{q\tilde{q}})}{(1-q\tilde{q})^2} \quad .$$

Proof. In the relation

$$P(s,k) = \left\{\frac{1 + q\tilde{q} - p\tilde{p}s - [(1-q\tilde{q})^2 - 2p\tilde{p}(1+q\tilde{q})s + (p\tilde{p})^2 s^2]^{1/2}}{2\tilde{q}}\right\}^k$$

make the substitutions

$$A = \frac{1+q\tilde{q}}{1-q\tilde{q}}$$

(17) $$x = \frac{p\tilde{p}}{1-q\tilde{q}}s \quad ,$$

$$y(x) = (2\tilde{q})^k(1-q\tilde{q})^{-k}P(s,k) \quad .$$

(5) It is interesting to note the similarity between this result and the corresponding one in the case of uniform spacing.

Then

$$y(x) = \left(A - x - \sqrt{1-2Ax+x^2}\right)^k .$$

Taking the logarithmic derivative,

(18) $$y' = \frac{ky}{\sqrt{1-2Ax+x^2}} ,$$

and

$$y'' = k\left[\frac{y'}{\sqrt{1-2Ax+x^2}} - \frac{y(x-A)}{(1-2Ax+x^2)^{3/2}}\right] .$$

Hence y satisfies the differential equation

(19) $$(1 - 2Ax + x^2)y'' = k^2y - (x - A)y' .$$

If we express y in the form

(20) $$y = (A - 1)^k \sum_{n=0}^{\infty} \frac{A_n}{n!}x^n ,$$

by equating coefficients in (19) we find that

(21) $$A_{n+2} = A(2n+1)A_{n+1} + (k^2-n^2)A_n, \quad n = 0,1,\dots .$$

Since

$$y(0) = [2\tilde{q}(1-q\tilde{q})]^{-k}P(0,k) = [2\tilde{q}(1-q\tilde{q})]^{-k}q^k = \left(\frac{2q\tilde{q}}{1-q\tilde{q}}\right)^k = (A-1)^k ,$$

it follows from (20) that

(22) $$A_o = 1 \quad .$$

Similarly, we see from (18) that

$$y'(0) = ky(0) = k(A-1)^k \quad ,$$

and hence from (20)

(23) $$A_1 = k \quad .$$

We now put

(24) $$A_n = \frac{C_n(n+k-1)!}{(k-1)!} \quad .$$

Then (21) becomes, after simplification,

(25) $$(n+k+1)C_{n+2} = A(2n+1)C_{n+1} + (k-n)C_n \quad ,$$

and the initial conditions (22) and (23) transform, respectively, into

$$C_o = 1$$

and

$$C_1 = 1 \quad .$$

To solve (25) we use the familiar method of Laplace. Set

$$C_n = \int_a^b t^{n-1}\mu(t)dt \quad ,$$

$$C_{n+1} = \int_a^b t^{n-1}[t\mu(t)]dt \quad ,$$

$$C_{n+2} = \int_a^b t^{n-1}[t^2\mu(t)]dt \quad ,$$

$$nCn = t^n\mu(t)\Big|_a^b - \int_a^b t^{n-1}[t\mu'(t)]dt \quad ,$$

$$(n+1)C_{n+1} = t^{n+1}\mu(t)\Big|_a^b - \int_a^b t^{n-1}[t^2\mu'(t)]dt \quad ,$$

and

$$(n+2)C_{n+2} = t^{n+2}\mu(t)\Big|_a^b - \int_a^b t^{n-1}[t^3\mu'(t)]dt \quad ,$$

where a, b, and $\mu(t)$ are to be determined. With these substitutions (25) takes the form

$$(26) \qquad \int_a^b t^{n-1}\left\{(t^3-2At^2+t)\mu'(t) - [(k-1)t^2+At-k]\mu(t)\right\}dt - t^n(t^2-2At+1)\mu(t)\Big|_a^b = 0 \quad .$$

To satisfy (26) it suffices to determine $\mu(t)$ so that the coefficient of t^{n-1} is 0 and then to choose a and b so that the integrated term is 0. Accordingly μ satisfies the differential equation

$$(27) \qquad \frac{\mu'}{\mu} = \frac{(k-1)t^2 + At-k}{t^3-2At^2+t} = -\frac{k}{t} + (k-\frac{1}{2})\frac{-2A+2t}{1-2At+t^2} \quad .$$

Solving (27) we obtain

$$\mu(t) = K\frac{(1-2At+t^2)^{k-\frac{1}{2}}}{t^k} \quad .$$

The equation for the integrated term,

$$(28) \qquad \left.\frac{t^n(t^2-2At+1)^{k+\frac{1}{2}}}{t^k}\right|_a^b = 0 \quad ,$$

is satisfied if a and b are roots of the quadratic equation

$$t^2 - 2At + 1 = 0 \quad .$$

If $n > k$, we can also choose $a = 0$. Thus, for $n > k$, (28) has distinct independent solutions

$$a = 0, \qquad b = A - \sqrt{A^2-1}$$

and

$$a = A - \sqrt{A^2-1} \; , \; b = A + \sqrt{A^2-1} \quad .$$

Hence, if $n > k$,

$$(29) \qquad C_n = K_1\int_0^{r_1} t^{n-k-1}(1-2At+t^2)^{k-\frac{1}{2}}dt +$$

$$K_2\int_{r_1}^{r_2} t^{n-k-1}[-(1-2At+t^2)]^{k-\frac{1}{2}}dt \quad ,$$

where

$$r_1 = A - \sqrt{A^2-1} \quad ,$$

$$r_2 = A + \sqrt{A^2-1} \quad ,$$

and K_1, K_2 are constants independent of n. (The negative sign is used in the second integral to make the term in brackets positive in the interval of integration.) Noting that

$$\int_0^{r_2} t^{n-k-1}|1-2At+t^2|^{k-\frac{1}{2}}dt = \int_0^{r_1} t^{n-k-1}(1-2At+t^2)^{k-\frac{1}{2}}dt$$

$$+\int_{r_1}^{r_2} t^{n-k-1}[-(1-2At+t^2)]^{k-\frac{1}{2}}dt \quad ,$$

we re-write (29) in the form

$$(30) \qquad c_n = \ell_1\int_0^{r_1} t^{n-k-1}|1-2At+t^2|^{k-\frac{1}{2}}dt + \ell_2\int_0^{r_2} t^{n-k-1}|1-2At+t^2|^{k-\frac{1}{2}}dt,$$

where ℓ_1 and ℓ_2 are constants independent of n.

The asymptotic behavior of these integrals as $n \longrightarrow \infty$ is revealed by making the transformation

$$t = r_i e^{-u} \quad , \quad i = 1,2 \; .$$

Then (30) becomes

$$c_n = \ell_1 r_1^{n-\frac{1}{2}} \int_0^\infty e^{-(n-k)u} |1-e^{-u}|^{k-\frac{1}{2}} |r_2 - r_1 e^{-u}|^{k-\frac{1}{2}} du$$
$$+ \ell_2 r_2^{n-\frac{1}{2}} \int_0^\infty e^{-(n-k)u} |1-e^{-u}|^{k-\frac{1}{2}} |r_1 - r_2 e^{-u}|^{k-\frac{1}{2}} du.$$

Let

$$F_i(u) = |1-e^{-u}|^{k-\frac{1}{2}} |r_j - r_i e^{-u}|^{k-\frac{1}{2}}$$

and

$$f_i(n) = r_i^{n-\frac{1}{2}} \int_0^\infty e^{-(n-k)u} F_i(u) du \quad ,$$

where $i = 1,2$, and $j = j(i) = \begin{cases} 2 \text{ if } i = 1, \\ 1 \text{ if } i = 2 \end{cases}$. Since

$$F_i(u) \sim u^{k-\frac{1}{2}} |r_j - r_i e^{-u}|^{k-\frac{1}{2}}$$

as $u \longrightarrow 0$, it follows from the familiar Abelian theorem for Laplace transforms[6] that as $n \longrightarrow \infty$

$$f_i(n) \sim \frac{r_i^{n-\frac{1}{2}} |r_j - r_i|^{k-\frac{1}{2}} \Gamma(k+\frac{1}{2})}{(n-k)^{k+\frac{1}{2}}} \sim \frac{r_i^{n-\frac{1}{2}} |r_j - r_i|^{k-\frac{1}{2}} \Gamma(k+\frac{1}{2})}{n^{k+\frac{1}{2}}} .$$

Moreover, since $r_1 < r_2$,

$$f_1(n) = o(f_2(n)) .$$

(6) Cf [7], P 181, Theorem 1, Corollary 1a.

Hence,

$$(31) \qquad c_n \sim \frac{\ell_1 r_1^{\,n-\frac{1}{2}} |r_2 - r_1|^{k-\frac{1}{2}} \Gamma(k+\frac{1}{2})}{n^{k+\frac{1}{2}}} .$$

Transforming back to the original variables (see (17), (20), (24), and (31)), we have

$$p(n,k) = \left(\frac{1-q\tilde{q}}{2\tilde{q}}\right)^k (A-1)^k \left(\frac{p\tilde{p}}{1-q\tilde{q}}\right)^n \frac{(n+k-1)!}{n!(k-1)!} c_n \sim \frac{Cr^n}{n^{3/2}} ,$$

where

$$r = \frac{r_1 p\tilde{p}}{1-q\tilde{q}} = \frac{p\tilde{p}(1+q\tilde{q}-2\sqrt{q\tilde{q}})}{(1-q\tilde{q})^2}\,^{(7)} .$$

This completes the proof.

---

(7) It is easy to verify that $r < 1$. Thus,

$$r = \frac{r_1 p\tilde{p}}{1-q\tilde{q}} < \frac{r_2 p\tilde{p}}{1-q\tilde{q}} = r_2\left(\frac{1+q\tilde{q}}{1-q\tilde{q}} - \frac{q+\tilde{q}}{1-q\tilde{q}}\right) = r_2\left[A - \frac{q+\tilde{q}}{2}(A+1)\right]$$

$$< r_2\left[A - \sqrt{q\tilde{q}}(A+1)\right] = r_2\left[A - \sqrt{\frac{A-1}{A+1}}(A+1)\right]$$

$$= r_2(A - \sqrt{A^2-1}) = r_1 r_2 = 1.$$

6. Remarks on the Continuous Case

It is interesting to note that results for the case when time is treated as a continuous variable can be obtained by an appropriate passage to the limit in the corresponding discrete formuli. To this end make a change of scale so that the discrete process takes place on the lattice points $\{mh,\ m = 0,1,\ldots\}$. Suppose the first element in the queue is at a distance kh from the absorbing barrier 0, and the distance between consecutive elements is dh. Then, as the analysis in §4.1 shows,

$$p(n,k) = \frac{k(k+nd-1)^{(n-1)}}{n!} p^n q^{k+n(d-1)} .$$

Now let α, T, and ε be positive constants, and, as in the passage from Bernoulli trials to the Poisson distribution, let $h \longrightarrow 0$, $p \longrightarrow 0$, $k \longrightarrow \infty$, and $d \longrightarrow \infty$ so that $kh \sim T$, $dh \sim \varepsilon$ and $p \sim \alpha h$. Then

$$p(n,k) \sim \frac{T(T+n\varepsilon)^{n-1}\alpha^n e^{-\alpha(T+n\varepsilon)}}{n!} ,$$

which agrees with the result in [3], where the service time is exponentially distributed, the initial element is at a distance T from 0, and the spacing is uniform at intervals ε.

If, in the discrete model, the spacing is geometric, let $\tilde{p} \sim \beta h$, where $\beta > 0$. Then, for the generating function $P(x,k)$ we obtain, in agreement with [3], the asymptotic formula

$$P(x,k) \sim e^{-\frac{T}{2}[(\alpha-\beta) + \sqrt{(\alpha+\beta)^2-4\alpha\beta x}]} .$$

Formuli for the moments of the number of elements served before termination can be obtained in the same way.

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